Spherically symmetric random walks. I. Representation in terms of orthogonal polynomials

Carl M. Bender
Department of Physics, Washington University, St. Louis, Missouri 63130

Fred Cooper
Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545

Peter N. Meisinger
Department of Physics, Washington University, St. Louis, Missouri 63130

(Received 6 June 1995; revised manuscript received 8 February 1996)

It is shown that, in general, a connection exists between orthogonal polynomials and semibounded random walks. This connection allows one to view a random walk as taking place on the set of integers that index the orthogonal polynomials. An illustration is provided by the case of spherically symmetric random walks. The correspondence between orthogonal polynomials and random walks enables one to express random-walk probabilities as weighted inner products of the polynomials. This correspondence is exploited to construct and analyze spherically symmetric random walks in D-dimensional space, where D is *not* restricted to be an integer. Such random walks can be described in terms of Gegenbauer (ultraspherical) polynomials. For example, Legendre polynomials can be used to represent the special case of two-dimensional spherically symmetric random walks. The weighted inner-product representation is used to calculate exact closed-form spatial and temporal moments of the probability distribution associated with the random walk. The polynomial representation of spherically symmetric random walks is then used to calculate the two-point Green's function for a rotationally symmetric free scalar quantum field theory. [S1063-651X(96)05606-1]

PACS number(s): 05.40.+j, 04.60.Nc, 02.90.+p

I. INTRODUCTION

Random walks on D-dimensional hypercubic lattices have been studied in great detail; see, for example [1,2], and the references therein. In two recent papers [3,4] we proposed and analyzed another kind of D-dimensional random walk that is well defined even when D is noninteger. This random walk takes place on a spherical lattice consisting of an infinite set of concentric nested spheres of radii R_n, $n=1,2,3,\ldots$. We define *region* n to be the volume lying between R_{n-1} and R_n, with the central region, region 1, being the volume inside R_1. If the random walker occupies region n at time t, then at time $t+1$ the random walker must move out to region $n+1$ with probability $P_{\rm out}(n)$ or in to region $n-1$ with probability $P_{\rm in}(n)$. The probabilities of moving out and in are in proportion to the hyperspherical surface areas bounding region n. Let $S_D(R)$ represent the surface area of a D-dimensional hypersphere

$$S_D(R)=\frac{2\pi^{D/2}}{\Gamma(D/2)}\,R^{D-1}.$$

We then take (for $n>1$)

$$P_{\rm out}(n)=\frac{S_D(R_n)}{S_D(R_n)+S_D(R_{n-1})}=\frac{R_n^{D-1}}{R_n^{D-1}+R_{n-1}^{D-1}} \quad (1.1)$$

and

$$P_{\rm in}(n)=\frac{S_D(R_{n-1})}{S_D(R_n)+S_D(R_{n-1})}=\frac{R_{n-1}^{D-1}}{R_n^{D-1}+R_{n-1}^{D-1}}. \quad (1.2)$$

For the special case $n=1$ we define

$$P_{\rm out}(1)=1,\quad P_{\rm in}(1)=0. \quad (1.3)$$

Note that probability is conserved because the total probability of the random walker moving out or in is *unity*:

$$P_{\rm out}(n)+P_{\rm in}(n)=1. \quad (1.4)$$

To describe a random walk on this lattice we introduce the notation $C_{n,t;m}$, which represents the probability that a random walker, initially in region m at time $t=0$, will be found in region n at time t. The probability $C_{n,t;m}$ satisfies the partial difference equation

$$C_{n,t;m}=P_{\rm in}(n+1)C_{n+1,t-1;m}+P_{\rm out}(n-1)C_{n-1,t-1;m} \quad (n\geqslant 2), \quad (1.5)$$

$$C_{1,t;m}=P_{\rm in}(2)C_{2,t-1;m}, \quad (1.6)$$

and the initial condition

$$C_{n,0;m}=\delta_{n,m}. \quad (1.7)$$

The random walk described above has the advantage that the quantity $C_{n,t;m}$ is a meaningful probability for *all* real values of the spatial dimension D; that is, for all times t, the inequality

$$0\leqslant C_{n,t;m}\leqslant 1$$

holds. This result is in stark contrast with the random walk as it is conventionally defined on a hypercubic lattice [5]. For example, on a D-dimensional hypercubic lattice, the probability that a random walker who is initially at the origin **0** will again be found at the origin at $t=2$ is

$$C_{0,2;0}=\frac{1}{2D},$$

which is greater than unity for $D<\frac{1}{2}$; the probability that a random walker who is initially at the origin **0** will again be found at the origin at $t=4$ is

$$C_{0,4;0}=\frac{6D-3}{8D^3},$$

which is negative for $D<\frac{1}{2}$.

In this paper we present an array of results concerning D-dimensional random walks. Specifically, we consider random walks that are defined by the partial difference equations (1.5)–(1.7). We show in Sec. II that there is a natural one-to-one correspondence between the probabilities $C_{n,t;m}$ that describe a random walk on a lattice consisting of regions n, $n=1,2,3,\ldots$, and a set of orthogonal polynomials $\{Q_{n-1}(x)\}$, $n=1,2,3,\ldots$. This set of polynomials is uniquely determined by the functions $P_{\rm in}(n)$ and $P_{\rm out}(n)$ in Eqs. (1.5)–(1.7). There is a simple expression for $C_{n,t;m}$ in terms of these polynomials. In general, one can view a random walk on the regions n as a random sequence of raising and lowering operators applied to the set of polynomials $\{Q_{n-1}(x)\}$. (Although not discussed in this paper, this correspondence between polynomials and random walks extends to multidimensional random walks and multi-index systems of orthonormal functions.)

If we take evenly spaced concentric spheres ($R_n=n$), we find that for the special cases $D=0,1,2$, the polynomials $\{Q_{n-1}(x)\}$ associated with $P_{\rm in}(n)$ and $P_{\rm out}(n)$ in Eqs. (1.1)–(1.3) are standard [6,7] classical polynomials: Gegenbauer polynomials for $D=0$, Chebyshev polynomials for $D=1$, and Legendre polynomials for $D=2$. However, for all other values of D the polynomials have not been previously studied and are not found in any of the usual treatments of orthogonal polynomials. While we can generate these polynomials, we have not been able to determine their general mathematical properties, such as their weight function and interval of orthogonality.

In Sec. III we modify the form of $P_{\rm in}(n)$ and $P_{\rm out}(n)$ in Eqs. (1.1)–(1.3) by replacing these functions with their large-n asymptotic behaviors. The polynomials that we now obtain are well-known classical polynomials (ultraspherical polynomials) *for all D*. This allows us to find closed-form expressions for the probabilities $C_{n,t;m}$ for all values of D.

Taking the probabilities in Sec. III, we then calculate in Sec. IV extraordinarily simple, closed-form, analytic expressions for the probability of a random walker eventually returning to the region from which the walker started, the expected time for the walker to return to the initial region, and other space and time moments of the probability distribution $C_{n,t;m}$. (In contrast, in Ref. [4], after heavy analysis we were only able to obtain asymptotic approximations for these moments.) We also find that for integer D these moments exhibit the qualitative features (e.g., Polya's theorem) of random walks on D-dimensional hypercubic lattices.

Our long-range objective in studying D-dimensional random walks is to understand critical behavior in quantum field theory. We would like to understand, for example, the transition that occurs when a self-interacting scalar ϕ^4 quantum field theory in space-time dimension $D<4$ becomes a free quantum field theory for $D>4$. One possible approach to such a problem would be to formulate a quantum field theory in terms of random walks [5,8]. However, if we do so on a hypercubic lattice, it is not possible to study these random walks except for integer values of D, as we have discussed above. As a result, we cannot use a hypercubic lattice to examine the behavior of a quantum field theory near $D=4$. Thus we are motivated to investigate alternative kinds of random walks that may be consistently defined for *all real D*. Critical behavior has already been observed in a two-dimensional spherically symmetric random-walk model [9]. In the next two papers in this series [10,11] we study this critical behavior as a continuous function of D for all $D>0$ in the context of birth and death models. We also show that polymers adhering to D-dimensional curved surfaces exhibit universal critical behavior.

Of course, a quantum field theory that is developed from a spherically symmetric random walk will itself be spherically symmetric. Such a theory is physically unacceptable because it violates causality. Nevertheless, the critical behavior that is observed in such a theory may well be a universal function of D and, at the very least, such a theory may provide some clues as to how a scalar quantum field theory can go from interacting to noninteracting at $D=4$. In Sec. V we carry out some preliminary investigations of spherically symmetric quantum field theory. Specifically, we use the machinery of spherically symmetric random walks that is developed in Secs. II–IV to obtain the free two-point Green's function of a rotationally symmetric scalar quantum theory.

II. CONNECTION BETWEEN POLYNOMIALS AND RANDOM WALKS

In this section we propose and discuss the following quadrature solution to partial difference equations of the type (1.5)–(1.7):

$$C_{n,t;m}=v_{n-1}\int_{-1}^{1}dx\ w(x)x^tQ_{n-1}(x)Q_{m-1}(x), \quad (2.1)$$

where $\{Q_n(x)\}$, $n=0,1,2,\ldots$, is a set of polynomials orthogonal with respect to $w(x)$ on the interval $-1\leqslant x\leqslant 1$ and $\{v_n\}$, $n=0,1,2,\ldots$, is a sequence of positive numbers. This proposed solution can in part be motivated by observing that the standard orthogonal polynomials [6] also satisfy three-term recursion relations. Further, differential equations similar to the difference equations satisfied by $C_{n,t;m}$ are encountered in queuing theory [12] and solutions to special cases of these differential equations are found in terms of standard orthogonal polynomials.

The form of (2.1) incorporates the initial condition (1.7) in a natural way. We simply choose to *normalize* the set of polynomials $\{Q_n(x)\}$ so that

$$\int_{-1}^{1}dx\ w(x)Q_n(x)Q_m(x)=\frac{1}{v_n}\ \delta_{n,m}. \quad (2.2)$$

With this choice of normalization we see that at $t=0$ (1.7) follows immediately from the above statement of orthogonality:

$$C_{n,0;m}=v_{n-1}\int_{-1}^{1}dx\ w(x)Q_{n-1}(x)Q_{m-1}(x)=\delta_{n,m}.$$

We now demand that the set of polynomials $\{Q_n(x)\}$ obey the recursion relation

$$P_{\mathrm{in}}(n+1)v_nQ_n(x)=v_{n-1}xQ_{n-1}(x)-P_{\mathrm{out}}(n-1)v_{n-2}Q_{n-2}(x)\quad(n\geqslant 2)\tag{2.3}$$

and the initial conditions

$$Q_0(x)=1\tag{2.4}$$

and

$$Q_1(x)=\frac{v_0}{P_{\mathrm{in}}(2)v_1}x.\tag{2.5}$$

The partial difference equations (1.5)–(1.7) for the probabilities $C_{n,t;m}$ are automatically satisfied so long as Eqs. (2.3)–(2.5) hold.

We will now show that

$$Q_n(1)=1\tag{2.6}$$

for all $n\geqslant 0$. This interesting property is a consequence of the conservation of probability; namely, the probability of finding the random walker *somewhere* on the lattice at an arbitrary time t is unity:

$$\sum_{n=1}^{\infty}C_{n,t;m}=1.\tag{2.7}$$

To establish Eq. (2.7) we merely sum Eqs. (1.5)–(1.7) over all $n\geqslant 1$, using Eqs. (1.3), (1.4), and (1.7). Assuming that the sum

$$f(x)\equiv\sum_{n=0}^{\infty}v_nQ_n(x)$$

exists in the space of distributions, we substitute the expression for $C_{n,t;m}$ in Eq. (2.1) into Eq. (2.7) to obtain

$$\int_{-1}^{1}dx\ w(x)x^tQ_{m-1}(x)f(x)=1.\tag{2.8}$$

Next, we compute the function $f(x)$ directly from the recursion relation (2.3) by summing over all $n\geqslant 2$, using Eqs. (2.4) and (2.5). We obtain the following equation for $f(x)$:

$$(1-x)f(x)=0.$$

The solution to this equation is a generalized function

$$f(x)=\alpha\delta(x-1),\tag{2.9}$$

where $\delta(s)$ is the Dirac delta function and α is a constant. Substituting Eq. (2.9) into Eq. (2.8) gives the condition that $Q_{m-1}(1)$ is a constant independent of m for all $m\geqslant 1$. Finally, from the conditions (2.4) and (2.5), we conclude that this constant is 1,

$$Q_m(1)=1,\tag{2.10}$$

and we therefore obtain Eq. (2.6).

The result (2.6) enables us to find a simple formula for the set of numbers $\{v_n\}$. We let $x=1$ in the recursion relation (2.3) to obtain

$$P_{\mathrm{in}}(n+1)v_n=v_{n-1}-P_{\mathrm{out}}(n-1)v_{n-2}\quad(n\geqslant 2)\tag{2.11}$$

and in the initial condition (2.5) to obtain

$$v_1=\frac{v_0}{P_{\mathrm{in}}(2)}.\tag{2.12}$$

The unique solution to Eq. (2.11) that satisfies Eq. (2.12) is

$$v_n=v_0\prod_{k=1}^{n}\frac{P_{\mathrm{out}}(k)}{P_{\mathrm{in}}(k+1)}\quad(n\geqslant 1).\tag{2.13}$$

The value of v_0 is determined from the orthogonality condition (2.2) at $n=m=0$ and the initial condition (2.4):

$$v_0=\frac{1}{\int_{-1}^{1}dx\ w(x)}.$$

The result in Eq. (2.13) can be used to eliminate the numbers v_n from the recursion relation (2.3), giving a much simpler recursion relation for the polynomials $Q_n(x)$:

$$P_{\mathrm{out}}(n)Q_n(x)=xQ_{n-1}(x)-P_{\mathrm{in}}(n)Q_{n-2}(x)\quad(n\geqslant 2).\tag{2.14}$$

The initial conditions in Eqs. (2.4) and (2.5) also become much simpler:

$$Q_0(x)=1,\quad Q_1(x)=x.$$

This recursion relation generates polynomials that exhibit parity symmetry; that is, even-index polynomials are even functions and odd-index polynomials are odd functions: $Q_n(-x)=(-1)^nQ_n(x)$. From the orthogonality condition (2.2) one can then deduce that the weight function $w(x)$ is an even function of x. As a consequence, we see from the integral representation in Eq. (2.1) that an n versus t table of values of the probabilities $C_{n,t;m}$ has a checkerboard pattern with nonzero entries alternating with zero entries in both the n and t directions. Evidently, a random walker starting from the site m at $t=0$ can only reach a site n at time t if $n+m+t$ is even. This parity condition is a consequence of the original definition of our random walk in which the walker must move in or out on every step and may not remain in the same region.

It is interesting to examine some special cases of the polynomial solution for $C_{n,t;m}$ in Eq. (2.1). We consider the case of equally spaced spherical shells $R_n=n$ and look at some particular values of the dimension D with $P_{\mathrm{out}}(n)$ and $P_{\mathrm{in}}(n)$ given in Eqs. (1.1)–(1.3).

A. Special case $R_n = n$, $D = 1$

Here

$$P_{\mathrm{out}}(n) = \tfrac{1}{2}, \quad P_{\mathrm{in}}(n) = \tfrac{1}{2} \quad (n \geqslant 2) \tag{2.15}$$

and

$$P_{\mathrm{out}}(1) = 1, \quad P_{\mathrm{in}}(1) = 0. \tag{2.16}$$

The polynomials $\{Q_n(x)\}$ are the standard Chebyshev polynomials of the first kind [6]:

$$\mathscr{T}_0(x) = 1, \quad \mathscr{T}_1(x) = x, \quad \mathscr{T}_2(x) = 2x^2 - 1,$$

$$\mathscr{T}_3(x) = 4x^3 - 3x, \quad \mathscr{T}_4(x) = 8x^4 - 8x^2 + 1,$$

and so on. For these polynomials, $w(x) = 1/\sqrt{1-x^2}$, $v_n = 2/\pi$, $n \geqslant 1$; and $v_0 = 1/\pi$. The random walk probabilities in Eq. (2.1) are given by

$$C_{n,t;m} = \frac{2}{\pi} \int_{-1}^{1} dx \, \frac{1}{\sqrt{1-x^2}} \, x^t \mathscr{T}_{n-1}(x) \mathscr{T}_{m-1}(x) \quad (n \geqslant 2),$$

$$C_{1,t;m} = \frac{1}{\pi} \int_{-1}^{1} dx \, \frac{1}{\sqrt{1-x^2}} \, x^t \mathscr{T}_{m-1}(x),$$

which for $m = 1$ reduces to the particular solution

$$C_{n,n+2j-1;1} = \frac{(n+2j-1)!}{j!(n+j-1)!2^{n+2j-2}} \quad (n \geqslant 2),$$

$$C_{1,2t;1} = \frac{(2t)!}{t!t!2^{2t}},$$

given in Ref. [3].

B. Special case $R_n = n$, $D = 2$

Here

$$P_{\mathrm{out}}(n) = \frac{n}{2n-1}, \quad P_{\mathrm{in}}(n) = \frac{n-1}{2n-1}. \tag{2.17}$$

The polynomials $\{Q_n(x)\}$ are the standard Legendre polynomials [6]

$$\mathscr{P}_0(x) = 1, \quad \mathscr{P}_1(x) = x, \quad \mathscr{P}_2(x) = \tfrac{1}{2}(3x^2 - 1),$$

$$\mathscr{P}_3(x) = \tfrac{1}{2}(5x^3 - 3x), \quad \mathscr{P}_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3),$$

and so on. For these polynomials, $w(x) = 1$ and $v_n = 2n+1$. Thus the random-walk probabilities in Eq. (2.1) are given by

$$C_{n,t;m} = (2n-1) \int_{-1}^{1} dx \, x^t \mathscr{P}_{n-1}(x) \mathscr{P}_{m-1}(x),$$

which for $m = 1$ reduces to the particular solution

$$C_{n,n+2j-1;1} = \frac{(2n-1)(n+2j-1)!}{j!(2n+2j-1)!!2^j},$$

given in Ref. [3].

C. Special case $R_n = n$, $D = 0$

Here

$$P_{\mathrm{out}}(n) = \frac{n-1}{2n-1}, \quad P_{\mathrm{in}}(n) = \frac{n}{2n-1} \quad (n \geqslant 2)$$

and

$$P_{\mathrm{out}}(1) = 1, \quad P_{\mathrm{in}}(1) = 0.$$

The first few polynomials in the set $\{Q_n(x)\}$ are

$$Q_0(x) = 1, \quad Q_1(x) = x, \quad Q_2(x) = 3x^2 - 2,$$
$$Q_3(x) = \tfrac{1}{2}(15x^3 - 13x), \quad Q_4(x) = \tfrac{1}{6}(105x^4 - 115x^2 + 16), \tag{2.18}$$

and so on. For these polynomials we have chosen $v_n = 3(2n+1)/[4n(n+1)]$, $n \geqslant 1$, and $v_0 = \frac{3}{4}$. The random walk probabilities in Eq. (2.1) are then given by

$$C_{n,t;m} = \frac{3(2n-1)}{4n(n-1)} \int_{-1}^{1} dx \, w(x) x^t Q_{n-1}(x) Q_{m-1}(x) \quad (n \geqslant 2),$$

$$C_{1,t;m} = \frac{3}{4} \int_{-1}^{1} dx \, w(x) x^t Q_{m-1}(x).$$

The polynomials Eq. (2.18) are closely related to the standard Gegenbauer (ultraspherical) polynomials $\{\mathscr{C}_n^{(\alpha)}(x)\}$ with upper index $\alpha = 3/2$ [6]. These particular Gegenbauer polynomials satisfy the recursion relation

$$(n+1)\mathscr{C}_{n+1}^{(3/2)}(x) = (2n+3)x\mathscr{C}_n^{(3/2)}(x) - (n+2)\mathscr{C}_{n-1}^{(3/2)}(x) \quad (n > 0)$$

and the initial conditions $\mathscr{C}_0^{(3/2)}(x) = 1$ and $\mathscr{C}_1^{(3/2)}(x) = 3x$. These Gegenbauer polynomials are orthogonal on the interval $-1 \leqslant x \leqslant 1$ with respect to the weight function $w(x) = 1 - x^2$. The polynomial $Q_{n+1}(x)$ satisfies the same recursion relation as these Gegenbauer polynomial $\mathscr{C}_n^{(3/2)}(x)$. However, it is generated from different initial conditions. We have been able to show that the weight function $w(x)$ with respect to which the set of polynomials $\{Q_n(x)\}$ is orthogonal satisfies the integral equation

$$\int_{-1}^{1} dt \, \frac{w(t)}{1 - xt^2} = \frac{2\sqrt{x}}{(1-x)[\ln(1+\sqrt{x}) - \ln(1-\sqrt{x})]}.$$

We do not know a closed-form solution to this equation.

D. Special case $R_n = n$, $D = 3$

Now,

$$P_{\mathrm{out}}(n) = \frac{n^2}{2n^2 - 2n + 1}, \quad P_{\mathrm{in}}(n) = \frac{(n-1)^2}{2n^2 - 2n + 1}.$$

For this case we can calculate any finite number of polynomials $\{Q_n(x)\}$:

$$Q_0(x)=1,\quad Q_1(x)=x,\quad Q_2(x)=\tfrac{1}{4}(5x^2-1),$$

$$Q_3(x)=\tfrac{1}{36}(65x^3-29x),$$

$$Q_4(x)=\tfrac{1}{576}(1625x^4-1130x^2+81),$$

and so on. These polynomials *are* orthogonal and they satisfy the normalization constraint (2.10). However, for this value of D (and for all values of D other than $D=0,1,2$) these polynomials are not related to the standard classical polynomials that one can find in reference books. We are unable to determine analytically the weight function $w(x)$ with respect to which these polynomials are orthogonal. Thus the formal expression Eq. (2.1) for the probabilities $C_{n,t;m}$ is not very useful. In the next section we devise a random walk process for which we *can* determine the weight function and thus find in closed form physically realistic probabilities $C_{n,t;m}$ for all values of $D>0$.

III. RANDOM WALKS FOR ULTRASPHERICAL POLYNOMIALS

In this section we show how to modify the expressions for $P_{\rm out}(n)$ and $P_{\rm in}(n)$ in Eqs. (1.1)–(1.3) so that we are able to obtain analytic closed-form expressions for $C_{n,t;m}$ for all values of $D>0$ for the case of evenly spaced spherical shells $R_n=n$. The random-walk process examined in Sec. II D is too difficult to solve in closed form simply because the formulas for $P_{\rm out}(n)$ and $P_{\rm in}(n)$ in Eqs. (1.1)–(1.3) become much too complicated when D takes on values other than 0, 1, or 2.

As we will see, the polynomials generated by the recursion relation (2.14) belong to a set of well-known classical polynomials if we take the formulas for $P_{\rm out}(n)$ and $P_{\rm in}(n)$ to be bilinear functions of n of the general form

$$P_{\rm out}(n)=\frac{an+b}{cn+d}, \tag{3.1}$$

with $P_{\rm in}(n)=1-P_{\rm out}(n)$. Note that bilinear functions contain *three* arbitrary parameters. We fix these parameters as follows. First, we demand that the random walk be confined to the values of $n\geqslant 1$. To impose this condition we require that $P_{\rm out}(1)=1$ or, equivalently, that $P_{\rm in}(1)=0$. This fixes one parameter. Second, we demand that the large-n asymptotic behavior of $P_{\rm out}(n)$ in Eqs. (1.1) and (3.1) agree to order n. These two conditions above yield the unique choice

$$P_{\rm out}(n)=\frac{n+D-2}{2n+D-3},\quad P_{\rm in}(n)=\frac{n-1}{2n+D-3}. \tag{3.2}$$

By determining the arbitrary parameters in Eq. (3.1) at the two boundary points $n=1$ and ∞ we obtain a uniformly accurate approximation to $P_{\rm out}(n)$ and $P_{\rm in}(n)$ in Eqs. (1.1)–(1.3) for all $n\geqslant 1$. In fact, Eq. (3.2) agrees exactly with Eqs. (1.1)–(1.3) for $D=1$ [see Eqs. (2.15) and (2.16)] and $D=2$ [see Eq. (2.17)]. For other values of D, Eq. (3.2) continues to be a good approximation, as verified in Fig. 1, where we compare $P_{\rm in}(n)$ in Eq. (1.2) with $P_{\rm in}(n)$ in Eq. (3.2) for several values of D.

The requirement that Eqs. (1.1) and (3.1) agree to order n as $n\to\infty$ incorporates the crucial dependence upon the dimension D of space; to wit, as D increases, a random walker is more likely to move outward than to move inward. As we will see in Sec. IV, it is this bias that gives rise to Polya's theorem; this theorem states that for $0<D\leqslant 2$ a random walker returns to the starting point with probability 1, while for $D>2$ this probability is less than 1.

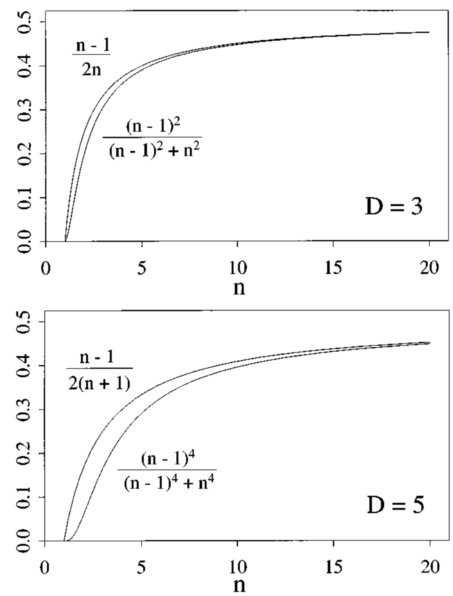

FIG. 1. Comparison between $P_{\rm in}(n)$ in Eq. (1.2) and the uniform approximation to $P_{\rm in}(n)$ in Eq. (3.2) for $D=3$ and 5. Note that the uniform approximation is exact at $D=1$ and 2.

Substituting the formulas above into Eq. (2.14) gives the recursion relation

$$(n+D-2)Q_n(x)=(2n+D-3)xQ_{n-1}(x) -(n-1)Q_{n-2}(x)\quad (n\geqslant 2). \tag{3.3}$$

Taking as initial conditions

$$Q_0(x)=1,\quad Q_1(x)=x,$$

we can easily use Eq. (3.3) to generate subsequent polynomials

$$Q_2(x)=\frac{1}{D}[(D+1)x^2-1],$$

$$Q_3(x)=\frac{1}{D}[(D+3)x^3-3x],$$

$$Q_4(x)=\frac{1}{D^2+2D}[(D^2+8D+15)x^4-(6D+18)x^2+3],$$

$$Q_5(x)=\frac{1}{D^2+2D}[(D^2+12D+35)x^5 -(10D+50)x^3+15x].$$

These polynomials are just the Gegenbauer (ultraspherical) polynomials [6] $\mathscr{C}_n^{(\alpha)}(x)$ normalized so that $Q_n(1)=1$:

$$Q_n(x)=\frac{n!\Gamma(D-1)}{\Gamma(n+D-1)}\,\mathscr{C}_n^{[(D-1)/2]}(x). \tag{3.4}$$

Gegenbauer polynomials are hypergeometric functions in the variable $(x-1)/2$; furthermore, since they are polynomials, the Taylor series for $Q_n(x)$ about $x=1$ terminates:

$$Q_n(x)=\sum_{j=0}^{n}\binom{n}{j}\frac{\Gamma(j+D+n-1)\Gamma(D/2)}{\Gamma(D+n-1)\Gamma(j+D/2)}\left(\frac{x-1}{2}\right)^j. \tag{3.5}$$

From the conventional theory of Gegenbauer polynomials [6] we immediately know the weight function with respect to which the polynomials $Q_n(x)$ are orthogonal:

$$w(x)=(1-x^2)^{(D-2)/2}. \tag{3.6}$$

Also, the normalization coefficients v_n in Eq. (2.2) are identified as

$$v_n=\frac{(2n+D-1)\Gamma(n+D-1)\Gamma[(D+1)/2]}{\sqrt{\pi}n!\Gamma(D/2)\Gamma(D)}. \tag{3.7}$$

Finally, we note that the polynomials $Q_n(x)$ satisfy the Sturm-Liouville eigenvalue differential equation

$$\left[(1-x^2)\frac{d^2}{dx^2}-Dx\frac{d}{dx}+n(n+D-1)\right]Q_n(x)=0$$

and the first-order difference-differential equation

$$\left[(1-x^2)\frac{d}{dx}+nx\right]Q_n(x)=nQ_{n-1}. \tag{3.8}$$

Now that we have identified explicitly the polynomials $Q_n(x)$, the weight function $w(x)$, and the normalization coefficients v_n, we can use the formula in Eq. (2.1) to calculate the moments of $C_{n,t;m}$ and obtain a physical description of our random walk.

IV. QUANTITATIVE DESCRIPTION OF THE RANDOM WALK

In this section we discuss the properties of the hyperspherical random walk introduced in Sec. III. We calculate the probability of eventually returning to the starting point of a random walk, the expected time of return, and various other moments of the random walk probabilities. As will be evident, the key advantage of this random walk is that all of these quantities can be calculated *in closed form.*

A. Probability of eventual return

In a physical description of a random walk the simplest and most natural question to ask is, What is the probability of eventually returning to the starting point? The probability that a random walker will eventually return to region m, given that the walker started in region m, is denoted $\Pi_m(D)$. To calculate $\Pi_m(D)$ we use generating function methods previously described [see Eq. (2.11) of Ref. [3]]; to wit,

$$\Pi_m(D)=1-\frac{1}{\sum_{t=0}^{\infty}C_{m,2t;m}}. \tag{4.1}$$

Our problem is now to evaluate the sum, which we denote by S_m, in Eq. (4.1). Using Eq. (2.1) with $Q_n(x)$ given in Eq. (3.4) we have

$$\begin{aligned}S_m&=\sum_{t=0}^{\infty}C_{m,2t;m}\\&=\sum_{t=0}^{\infty}v_{m-1}\int_{-1}^{1}dx(1-x^2)^{(D-2)/2}x^{2t}[Q_{m-1}(x)]^2.\end{aligned} \tag{4.2}$$

Note that this sum is divergent unless $D>2$. To verify this assertion we observe that the large-t asymptotic behavior of the integral in Eq. (4.2) is

$$\int_{-1}^{1}dx(1-x^2)^{(D-2)/2}x^{2t}[Q_{m-1}(x)]^2\sim\Gamma(D/2)t^{-D/2}\qquad(t\to\infty).$$

Thus

$$\Pi_m(D)=1\quad(0<D\leqslant 2). \tag{4.3}$$

When $D>2$ the sum S_m converges, and we begin the evaluation by interchanging the order of summation and integration:

$$S_m=v_{m-1}\int_{-1}^{1}dx(1-x^2)^{(D-4)/2}[Q_{m-1}(x)]^2.$$

We evaluate this integral exactly using the recursion relation (3.3) and the difference-differential equation (3.8). The result is

$$S_m=\frac{2m+D-3}{D-2}. \tag{4.4}$$

Substituting into Eq. (4.1) gives

$$\Pi_m(D)=\frac{2m-1}{2m+D-3}\quad(D>2). \tag{4.5}$$

This result is exact for all m and D [13]. The probability in Eqs. (4.3) and (4.5) confirms that Polya's theorem holds for this model of a random walk, regardless of the region in which the walk begins.

For a hypercubic lattice the probability of eventually returning to the starting point of a random walk is given in terms of an integral [see Eq. (2.12) of Ref. [3]]

$$\Pi(D)=1-\frac{1}{\int_0^{\infty}dt\,e^{-t}[I_0(t/D)]^D},$$

where $I_0(x)$ is the modified Bessel function. Unlike the random walk discussed in this paper, when $D>2$, $\Pi(D)$ cannot be given in closed form (except for the special case $D=3$).

However, the asymptotic expansion of $\Pi(D)$ for large D is known [see Eq. (2.15) of Ref. [3]]:

$$\Pi(D) \sim \frac{1}{2D} + \frac{1}{2D^2} + \cdots \quad (D \to \infty).$$

Note that for large D, the probability of returning to the starting point of a random walk falls off algebraically like $1/D$ in both models. In contrast, for the hyperspherical random walk discussed in Ref. [3], the probability function $\Pi_1(D) = 1 - 1/\zeta(D-1)$ falls off *exponentially* like 2^{1-D} for large D. As functions of D, the hypercubic $\Pi(D)$ and the hyperspherical $\Pi_m(D)$ discussed here both exhibit cusps at $D=2$.

Observe that for large m, $\Pi_m(D)$ approaches 1. This happens because the available entropy for the random walk becomes constant; at large radius a sphere looks locally like a plane. Indeed, as $n \to \infty$, the recursion relation (3.3) approaches that of a one-dimensional random walk for which $P_{\rm out}(n) = P_{\rm in}(n) = \frac{1}{2}$.

B. Expected time of return

As explained in Ref. [4], the expected time of return $T_m(D)$ of a random walker who begins the walk in region m is obtained from the first moment of $C_{m,t;m}$:

$$T_m(D) = \frac{\Sigma_{t=0}^{\infty} 2tC_{m,2t;m}}{\Pi_m(D)(\Sigma_{t=0}^{\infty} C_{m,2t;m})^2}. \tag{4.6}$$

Again, using formulas (3.3) and (3.8) we can calculate the sums in Eq. (4.6) straightforwardly. We find that

$$T_m(D) = \begin{cases} \infty & (0 < D \leqslant 4) \\ \dfrac{2(D-2)[(2m-1)D + 2(m-1)(m-2)]}{(2m-1)D(D-4)}. & (D > 4) \end{cases}$$

Note that as D increases, $T_m(D)$ approaches 2, independent of the starting point m. This is because for very large dimension D, if a random walker does not return to the starting point on the second step, the random walker will *never* return; as $D \to \infty$ the entropy for moving outward dominates the walk. However, for fixed D as m increases $T_m(D)$ diverges. This is because for large m the D-dimensional walk approaches a one-dimensional walk for which the expected time of return is infinite.

C. Higher temporal moments

In general, all temporal moments can be calculated in closed form. The pth temporal moment $\Sigma_{t=0}^{\infty} t^p C_{m,2t;m}$ is a rational function of D and m whose complexity increases with p. The sum defining the pth temporal moment converges when $D > 2p+2$ and diverges when $D \leqslant 2p+2$. We list the first four temporal moments below [note that the zeroth moment S_m is already given in Eq. (4.4)]

$$\sum_{t=0}^{\infty} C_{m,2t;m} = S_m = \frac{(2m+D-3)M_0}{D-2},$$

$$\sum_{t=0}^{\infty} tC_{m,2t;m} = \frac{(2m+D-3)M_1}{(D-4)(D-2)D},$$

$$\sum_{t=0}^{\infty} t^2 C_{m,2t;m} = \frac{(2m+D-3)M_2}{(D-6)(D-4)(D-2)D(D+2)},$$

$$\sum_{t=0}^{\infty} t^3 C_{m,2t;m} = \frac{(2m+D-3)M_3}{(D-8)(D-6)(D-4)(D-2)D(D+2)(D+4)},$$

where

$$M_0 = 1,$$

$$M_1 = (2m-1)D + 2(m-1)(m-2),$$

$$M_2 = (2m-1)D^3 + 2(7m^2 - 13m + 7)D^2 + 4(m-1)(6m^2 - 17m + 16)D + 12(m-1)(m-2)(m^2 - 3m + 4),$$

$$\begin{aligned} M_3 = {} & (2m-1)D^5 + 2(19m^2 - 29m + 15)D^4 + 2(96m^3 \\ & - 296m^2 + 386m - 173)D^3 + 4(99m^4 - 486m^3 \\ & + 1093m^2 - 1184m + 477)D^2 + 4(90m^5 - 621m^4 \\ & + 2040m^3 - 3683m^2 + 3414m - 1252)D + 24(m-1) \\ & \times (m-2)(5m^4 - 30m^3 + 97m^2 - 156m + 104). \end{aligned}$$

D. Spatial moments

The kth spatial moment of a random walk is defined as a weighted average over the probabilities $C_{n,t;m}$:

$$\langle R^k \rangle_t \equiv \sum_{n=1}^{\infty} n^k C_{n,t;m}. \tag{4.7}$$

Note that in general $\langle R^k \rangle_t$ depends on the starting point m of the random walk. We have suppressed the argument m because, as we will see, the leading asymptotic behavior of $\langle R^k \rangle_t$ as $t \to \infty$ and the first correction to this behavior are independent of m. (The second correction does depend on m.)

We have found an exact expression for $\langle R^k \rangle_t$ for all values of t for the special case $m=1$:

$$\langle R^k \rangle_{2t} = (2t+1)^k + \sum_{r=1}^{t} \sum_{s=r+1}^{t+1} (-1)^{r+s} \times f(2r-1, 2s-1, 2t, k) \tag{4.8}$$

and

$$\langle R^k \rangle_{2t+1} = (2t+2)^k + \sum_{r=1}^{t} \sum_{s=r+1}^{t+1} (-1)^{r+s} f(2r, 2s, 2t+1, k), \tag{4.9}$$

where

$$f(x,y,t,k)=\frac{(x-y)\Gamma(x+y-1)\Gamma(t+1)(x^k-y^k)}{2^t\Gamma(x)\Gamma(y)\Gamma\left(\frac{x+y}{2}\right)\Gamma\left(\frac{t-x+3}{2}\right)\Gamma\left(\frac{t-y+3}{2}\right)(D+x+y-3)}. \tag{4.10}$$

This formula has the virtue that the D dependence is very simple; the parameter D occurs just once in the denominator of f in Eq. (4.10). Furthermore, for the special case of the zeroth moment, setting $k=0$ in Eq. (4.8) or (4.9) immediately gives the result $\langle R^0\rangle_t=1$, which states that probability is conserved. For $k>1$ this formula is inherently complicated. It is not easy to determine the asymptotic behavior of $\langle R^k\rangle_t$ for large t from Eqs. (4.8) or (4.9) because terms in the double sum oscillate in sign.

To find the asymptotic behavior of $\langle R^k\rangle_t$ as $t\to\infty$ we use generating-function techniques. We rewrite Eq. (4.7) as a derivative operator applied k times to a power series:

$$\langle R^k\rangle_t=\lim_{z\to 1}\left(z\,\frac{d}{dz}\right)^k\sum_{n=1}^{\infty}z^nC_{n,t;m}. \tag{4.11}$$

Next, we substitute into Eq. (4.11) the integral representation for the probability $C_{n,t;m}$ in Eq. (2.1) and use Eq. (3.6). We obtain

$$\langle R^k\rangle_t=\lim_{z\to 1}\int_{-1}^{1}dx(1-x^2)(D-2)/2x^tQ_{m-1}(x)\times\left(z\,\frac{d}{dz}\right)^k\left[z\sum_{n=0}^{\infty}z^nv_nQ_n(x)\right]. \tag{4.12}$$

It is convenient to use the expression for v_n in Eq. (3.7) and the recursion relation (3.3) for $Q_n(x)$ to evaluate the sum in Eq. (4.12):

$$\sum_{n=0}^{\infty}z^nv_nQ_n(x)=\frac{\Gamma\left(\frac{D+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{D}{2}\right)}(1-z^2)\times(1-2xz+z^2)^{-(D+1)/2}.$$

We are interested in the behavior of the resulting integral as $t\to\infty$. By Laplace's method this integral is dominated by values of x near 1 in this limit. Thus, for fixed m Eq. (2.10) implies that we may replace $Q_{m-1}(x)$ by 1 to leading order; we thus conclude that the leading asymptotic behavior of $\langle R^k\rangle_t$ is independent of m. To obtain higher-order terms in the asymptotic expansion we replace $Q_{m-1}(x)$ by the expansion in Eq. (3.5). A straightforward asymptotic analysis of the resulting integral gives the first few terms in the asymptotic expansion of $\langle R^k\rangle_t$ for large t with m fixed:

$$\langle R^k\rangle_t\sim\frac{\Gamma\left(\frac{D+k}{2}\right)}{\Gamma\left(\frac{D}{2}\right)}(2t)^{k/2}\left\{1-\frac{k(D-3)\Gamma\left(\frac{D+k-1}{2}\right)}{2\Gamma\left(\frac{D+k}{2}\right)}\times(2t)^{-1/2}+k\left[\frac{(m-1)(m+D-2)}{D}+\frac{3D^2k-18Dk+12D-2k^2+33k-28}{12(D+k-2)}\right](2t)^{-1}+O(t^{-3/2})\right\}\quad(t\to\infty). \tag{4.13}$$

Observe that the leading term in this asymptotic expansion is precisely the same as the result in Eq. (3.4) of Ref. [4] for the case of spherically symmetric random walks described by the probabilities in Eqs. (1.1)–(1.3) with $R_n=n$. The result in Eq. (4.13) is obtained directly and with considerably less effort than that in Ref. [4], where only the leading asymptotic behavior was obtained. Note that the first two terms in Eq. (4.13) are independent of the starting point m. To verify the accuracy of this asymptotic expansion we compare the first three partial sums of this series with the exact values of the moments obtained numerically at $t=1000$; this comparison is given in Tables I and II. In Table I we consider the kth moment for various values of k and D with $m=1$. In Table II we consider the first and second moments for various values of the starting point m with $D=2$.

From the asymptotic behavior in Eq. (4.13) with $k=2$ we can determine the Hausdorff dimension D_H of the random walk [4]. We find that

$$D_H=2$$

for all values of D. This result agrees with that obtained in [5] for a D-dimensional hyper-cubic lattice.

V. APPLICATION TO QUANTUM FIELD THEORY

One of our long-range goals in our study of D-dimensional random walks is a deeper understanding of D-dimensional quantum field theory. In particular, we are interested in how critical phenomena in such theories depend on the dimension of space-time. We are especially interested in how a ϕ^4 scalar field theory becomes free as $D\to 4$. We have already conducted several investigations of D-dimensional quantum-mechanical and field-theoretic systems [14–17]. In this section, as an elementary illustration of how to apply our work on D-dimensional random walks to quantum field theory, we use the random walk probabilities $C_{n,t;m}$ in Eq. (2.1) with the polynomials $Q_n(x)$ given in Eq.

TABLE I. Actual and predicted values of $\Sigma_{n=1}^{\infty} n^k C_{n,t;m}$ for $t=1000$ and $m=1$.

k	D	Actual	Leading behavior	With first correction	With second correction
1	1	$2.622\,502\times10^{1}$	$2.523\,133\times10^{1}$	$2.623\,133\times10^{1}$	$2.623\,133\times10^{1}$
	2	$4.013\,810\times10^{1}$	$3.963\,327\times10^{1}$	$4.013\,327\times10^{1}$	$4.013\,823\times10^{1}$
	3	$5.047\,526\times10^{1}$	$5.046\,265\times10^{1}$	$5.046\,265\times10^{1}$	$5.047\,527\times10^{1}$
	4	$5.987\,220\times10^{1}$	$5.944\,991\times10^{1}$	$5.894\,991\times10^{1}$	$5.897\,220\times10^{1}$
	5	$6.631\,717\times10^{1}$	$6.728\,353\times10^{1}$	$6.628\,353\times10^{1}$	$6.631\,718\times10^{1}$
2	1	$1.051\,450\times10^{3}$	$1.000\,000\times10^{3}$	$1.050\,463\times10^{3}$	$1.051\,463\times10^{3}$
	2	$2.040\,138\times10^{3}$	$2.000\,000\times10^{3}$	$2.039\,633\times10^{3}$	$2.040\,133\times10^{3}$
	3	$3.001\,000\times10^{3}$	$3.000\,000\times10^{3}$	$3.000\,000\times10^{3}$	$3.001\,000\times10^{3}$
	4	$3.943\,028\times10^{3}$	$4.000\,000\times10^{3}$	$3.940\,550\times10^{3}$	$3.943\,050\times10^{3}$
	5	$4.870\,366\times10^{3}$	$5.000\,000\times10^{3}$	$4.865\,433\times10^{3}$	$4.870\,433\times10^{3}$
3	1	$5.352\,671\times10^{4}$	$5.046\,265\times10^{4}$	$5.346\,265\times10^{4}$	$5.352\,573\times10^{4}$
	2	$1.219\,251\times10^{5}$	$1.188\,998\times10^{5}$	$1.218\,998\times10^{5}$	$1.219\,246\times10^{5}$
	3	$2.019\,011\times10^{5}$	$2.018\,506\times10^{5}$	$2.018\,506\times10^{5}$	$2.019\,011\times10^{5}$
	4	$2.914\,616\times10^{5}$	$2.972\,495\times10^{5}$	$2.912\,495\times10^{5}$	$2.914\,651\times10^{5}$
	5	$3.892\,603\times10^{5}$	$4.037\,012\times10^{5}$	$3.887\,012\times10^{5}$	$3.892\,731\times10^{5}$
4	1	$3.205\,902\times10^{6}$	$3.000\,000\times10^{6}$	$3.201\,851\times10^{6}$	$3.205\,851\times10^{6}$
	2	$8.237\,810\times10^{6}$	$8.000\,000\times10^{6}$	$8.237\,800\times10^{6}$	$8.237\,800\times10^{6}$
	3	$1.500\,000\times10^{7}$	$1.500\,000\times10^{7}$	$1.500\,000\times10^{7}$	$1.500\,000\times10^{7}$
	4	$2.342\,114\times10^{7}$	$2.400\,000\times10^{7}$	$2.340\,550\times10^{7}$	$2.342\,150\times10^{7}$
	5	$3.344\,349\times10^{7}$	$3.500\,000\times10^{7}$	$3.338\,520\times10^{7}$	$3.344\,520\times10^{7}$

(3.4) to calculate the Euclidean two-point Green's function of a D-dimensional free scalar quantum field theory having spherical symmetry. We will then verify our calculation by taking the spherical average of the two-point Green's function of a conventional translationally invariant (nonspherically symmetric) Euclidean field theory [18].

A. Derivation of spherically symmetric propagator from random-walk probabilities $C_{n,t;m}$

For this calculation we follow the standard recipe discussed in Ref. [5]. Specifically, we begin with the generating function $G(n,m,\lambda)$ for the temporal moments of the probabilities $C_{n,t;m}$:

$$G(n,m,\lambda)=\sum_{t=0}^{\infty}\lambda^t C_{n,t;m}.$$

Our objective is to find the continuum limit of this expression.

For definiteness we choose n, m, and t to be even: $n=2N$, $m=2M$, and $t=2T$. Also, without loss of generality, we take $N\geqslant M$. Substituting the formula for $C_{m,t;m}$ in Eq. (2.1) with $Q_n(x)$ given by ultraspherical polynomials in Eq. (3.4), and $w(x)$ in Eq. (3.6) and v_n in Eq. (3.7), we obtain

$$G(2N,2M,\lambda)=\frac{4N\Gamma^2\left(\frac{D-1}{2}\right)}{\pi M^{D-2}}\int_0^1 dx(1-x^2)^{(D-2)/2} \times\sum_{T=N-M}^{\infty}(x\lambda)^{2T}\mathscr{C}_{2N-1}^{[(D-1)/2]} \times(x)\mathscr{C}_{2M-1}^{[(D-1)/2]}(x). \tag{5.1}$$

TABLE II. Actual and predicted values of $\Sigma_{n=1}^{\infty} n^k C_{n,t;m}$ for $t=1000$ and $D=2$.

k	m	Actual	Leading behavior	With first correction	With second correction
1	1	$4.013\,810\times10^{1}$	$3.963\,327\times10^{1}$	$4.013\,327\times10^{1}$	$4.013\,823\times10^{1}$
	3	$4.019\,752\times10^{1}$	$3.963\,327\times10^{1}$	$4.013\,327\times10^{1}$	$4.019\,768\times10^{1}$
	5	$4.033\,597\times10^{1}$	$3.963\,327\times10^{1}$	$4.013\,327\times10^{1}$	$4.033\,639\times10^{1}$
	7	$4.055\,306\times10^{1}$	$3.963\,327\times10^{1}$	$4.013\,327\times10^{1}$	$4.055\,438\times10^{1}$
	9	$4.084\,814\times10^{1}$	$3.963\,327\times10^{1}$	$4.013\,327\times10^{1}$	$4.085\,163\times10^{1}$
2	1	$2.040\,138\times10^{3}$	$2.000\,000\times10^{3}$	$2.039\,633\times10^{3}$	$2.040\,133\times10^{3}$
	3	$2.046\,198\times10^{3}$	$2.000\,000\times10^{3}$	$2.039\,633\times10^{3}$	$2.046\,133\times10^{3}$
	5	$2.060\,336\times10^{3}$	$2.000\,000\times10^{3}$	$2.039\,633\times10^{3}$	$2.060\,133\times10^{3}$
	7	$2.082\,553\times10^{3}$	$2.000\,000\times10^{3}$	$2.039\,633\times10^{3}$	$2.082\,133\times10^{3}$
	9	$2.112\,848\times10^{3}$	$2.000\,000\times10^{3}$	$2.039\,633\times10^{3}$	$2.112\,133\times10^{3}$

Next, we perform the sum in Eq. (5.1):

$$G(2N,2M,\lambda)=\frac{4N\Gamma^2\left(\frac{D-1}{2}\right)}{\pi M^{D-2}}\int_0^1 dx(1-x^2)^{(D-2)/2}\times\frac{(x\lambda)^{2N-2M}}{1-x^2\lambda^2}\mathscr{C}_{2N-1}^{[(D-1)/2]}\times(x)\mathscr{C}_{2M-1}^{[(D-1)/2]}(x). \tag{5.2}$$

To prepare for taking the continuum limit we make use of the equivalence of Gegenbauer and Jacobi polynomials [6]:

$$\mathscr{C}_{2N-1}^{[(D-1)/2]}(x)=\frac{\Gamma(D/2)\Gamma(2N-2+D)}{\Gamma(D-1)\Gamma(2N-1+D/2)}\times\mathscr{P}_{2N-1}^{[(D-2)/2,(D-2)/2]}(x). \tag{5.3}$$

Substituting Eq. (5.3) into Eq. (5.2) and taking N and M large gives

$$G(2N,2M,\lambda)=2^{4-D}N^{D/2}M^{1-D/2}\int_0^1 dx(1-x^2)^{(D-2)/2}\times\frac{(x\lambda)^{2N-2M}}{1-x^2\lambda^2}\mathscr{P}_{2N-1}^{[(D-2)/2,(D-2)/2]}\times(x)\mathscr{P}_{2M-1}^{[(D-2)/2,(D-2)/2]}(x). \tag{5.4}$$

When N and M are large the integral in Eq. (5.4) is dominated by values of x near 1. Thus we make the change of variable $x=1-\epsilon^2s^2/2$, where ϵ is a small parameter:

$$G(2N,2M,\lambda)=4\epsilon^2N^{D/2}M^{1-D/2}\times\int_0^{\sqrt{2}/\epsilon}ds\,\frac{s[\lambda(1-\epsilon^2s^2/2)]^{2N-2M}}{1-\lambda^2(1-\epsilon^2s^2/2)^2}\times\left(\frac{\epsilon s}{2}\right)^{(D-2)/2}\mathscr{P}_{2N-1}^{[(D-2)/2,(D-2)/2]}\left(1-\frac{\epsilon^2s^2}{2}\right)\times\left(\frac{\epsilon s}{2}\right)^{(D-2)/2}\mathscr{P}_{2M-1}^{[(D-2)/2,(D-2)/2]}\left(1-\frac{\epsilon^2s^2}{2}\right). \tag{5.5}$$

We now make use of the following asymptotic limit for Jacobi polynomials [6]:

$$\lim_{\eta\to0}\left(\frac{\eta s}{2}\right)^{\alpha}\mathscr{P}_{1/\eta}^{(\alpha,\alpha)}\left(1-\frac{\eta^2s^2}{2}\right)=J_\alpha(s),$$

where $J_\alpha(s)$ is a Bessel function. Because N and M are large, we can use this asymptotic limit twice in Eq. (5.5):

$$G(2N,2M,\lambda)=4\epsilon^2N^{D/2}M^{1-D/2}\times\int_0^{\sqrt{2}/\epsilon}ds\,\frac{s[\lambda(1-\epsilon^2s^2/2)]^{2N-2M}}{1-\lambda^2(1-\epsilon^2s^2/2)^2}\times J_{(D-2)/2}(\epsilon(2N-1)s)J_{(D-2)/2}\times(\epsilon(2M-1)s). \tag{5.6}$$

We introduce the continuum variables r and r' by

$$\mu r=\epsilon(2N-1),\quad \mu r'=\epsilon(2M-1),$$

where μ is a mass parameter. Note that $r>r'$. Also, since $\epsilon\ll1$, we may replace the upper limit of integration in Eq. (5.6) by ∞ and simplify the integrand:

$$G(2N,2M,\lambda)=2\epsilon r^{D/2}(r')^{1-D/2}\mu\times\int_0^{\infty}ds\,\frac{s}{1-\lambda^2(1-\epsilon^2s^2)}J_{(D-2)/2}(\mu rs)\times J_{(D-2)/2}(\mu r's).$$

Finally, we make use of the Bessel function integral identity [19]

$$\int_0^\infty ds\,\frac{s}{s^2+c^2}J_\nu(as)J_\nu(bs)=I_\nu(bc)K_\nu(ac)\quad(a>b),$$

where I_ν and K_ν are modified Bessel functions. Taking, $\lambda^2\epsilon^2=1-\lambda^2$, we have

$$G(2N,2M,\lambda)=\frac{2\mu}{\epsilon}r^{D/2}(r')^{1-D/2}I_{(D-2)/2}(\mu r')\times K_{(D-2)/2}(\mu r). \tag{5.7}$$

Apart from a multiplicative normalization constant, the expression in Eq. (5.7) is the final result for the Euclidean propagator. Let $\mathscr{G}(r\to r')$ represent the spherically averaged amplitude for a free scalar particle of mass μ to propagate from some point on a sphere of radius r to some point on a sphere of radius r'. Note that this probability amplitude is not symmetric under the interchange of r and r'; when $D>1$ it is more likely for a particle to propagate from a sphere of smaller radius to a sphere of larger radius than for the reverse to occur. This is because the final state of the particle propagating to the larger sphere has a higher entropy. This asymmetry does not occur in translationally invariant theories. Our final, properly normalized, result for the propagator is

$$\mathscr{G}(r\to r')=(r')^{D-1}(rr')^{1-D/2}I_{(D-2)/2}(\mu r_<)\times K_{(D-2)/2}(\mu r_>),\tag{5.8}$$

where

$$r_>=\max\{r,r'\},\quad r_<=\min\{r,r'\}.$$

The normalization of the Green's function in Eq. (5.8) will be verified in Sec. V B.

The propagation asymmetry in Eq. (5.8) is a continuum manifestation of the directional bias that is present in spherically symmetric random walks. Note that $C_{n,t;m}$, the probability of walking from m to n [see Eq. (2.1)], is *not* a symmetric function of m and n. Rather, it is the function v_{n-1} in Eq. (3.7) multiplying a symmetric function of m and n. The function v_{n-1} represents the random-walk entropy associated with the volume of hyperspherical region n. The asymmetry in Eq. (5.8) is a direct consequence of the asymmetry in $C_{n,t;m}$.

B. Normalization of the two-point Green's function

The free propagator in momentum space for a D-dimensional translationally symmetric scalar field theory is

$$\tilde{\mathscr{G}}(\mathbf{k})=\frac{1}{k^2+\mu^2}.$$

To obtain the coordinate-space propagator $\mathscr{G}(\mathbf{r}-\mathbf{r}')$ we take the D-dimensional Fourier transform of the momentum-space propagator:

$$\mathscr{G}(\mathbf{r}-\mathbf{r}')=\frac{1}{(2\pi)^D}\int\frac{d^Dk}{k^2+\mu^2}e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}=\frac{1}{(2\pi)^{D/2}}(\mu/|\mathbf{r}-\mathbf{r}'|)^{(D-2)/2}\times K_{(D-2)/2}(\mu|\mathbf{r}-\mathbf{r}'|).\tag{5.9}$$

The coordinate-space propagator satisfies the Green's-function differential equation

$$(\nabla^2-\mu^2)\mathscr{G}(\mathbf{r}-\mathbf{r}')=\delta^{(D)}(\mathbf{r}-\mathbf{r}').$$

Let us calculate the amplitude for a particle at $\mathbf{r}$ to propagate *anywhere*. We obtain this amplitude by integrating $\mathscr{G}(\mathbf{r}-\mathbf{r}')$ with respect to $\mathbf{r}'$ over all space:

$$\int d^Dr'\,\mathscr{G}(\mathbf{r}-\mathbf{r}')=\frac{1}{\mu^2}.\tag{5.10}$$

We can now verify that $\mathscr{G}(r\to r')$ in Eq. (5.8) is properly normalized by calculating the amplitude for a particle at radius r to propagate to *any* radius:

$$\begin{aligned}\int_0^\infty dr'\,\mathscr{G}(r\to r')&=r^{1-D/2}K_{(D-2)/2}(\mu r)\times\int_0^r dr'(r')^{D/2}I_{(D-2)/2}(\mu r')\\&\quad+r^{1-D/2}I_{(D-2)/2}(\mu r)\int_r^\infty dr'(r')^{D/2}\times K_{(D-2)/2}(\mu r')\\&=\frac{r}{\mu^2}[I_{D/2}(r)K_{(D-2)/2}(r)+I_{(D-2)/2}(r)K_{D/2}(r)]=\frac{1}{\mu^2},\end{aligned}$$

where we have the used the Wronskian identity for modified Bessel functions. This result agrees with that in Eq. (5.10).

C. Continuum derivation of spherically symmetric propagator

In this subsection we derive the spherically symmetric propagator in Eq. (5.8) from the translationally symmetric propagator in Eq. (5.9) by taking an angular average. To obtain the angular average we let

$$|\mathbf{r}-\mathbf{r}'|=\sqrt{r^2+(r')^2-2rr'\cos\theta}.$$

We then expand the modified Bessel function in Eq. (5.9) as a series in terms of Gegenbauer polynomials

$$\frac{K_\nu(\mu|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|^\nu}=\Gamma(\nu)\left(\frac{1}{2}\mu^2rr'\right)^{-\nu}\sum_{n=0}^\infty(n+\nu)\mathscr{C}_n^{(\nu)}\times(\cos\theta)I_{n+\nu}(\mu r)K_{n+\nu}(\mu r')\qquad(r<r').$$

If we then integrate over the angle θ, only the $n=0$ term in the series survives and we obtain the result in Eq. (5.8). The fact that we obtain the same two-point Green's function directly from our random-walk model supports the validity of the uniform approximation for the probabilities $P_{\rm out}(n)$ and $P_{\rm in}(n)$ in Eq. (3.2).

ACKNOWLEDGMENTS

We thank the U.S. Department of Energy for Financial Support. One of us (P.N.M.) thanks the U.S. Department of Education for additional financial support.

[1] G. H. Weiss, *Aspects and Applications of the Random Walk* (North-Holland, Amsterdam, 1994), Chap. 5.

[2] B. D. Hughes, *Random Walks and Random Environments* (Clarendon, Oxford, 1995), Sec. 3.4.

[3] C. M. Bender, S. Boettcher, and L. R. Mead, J. Math. Phys. **35**, 368 (1994).

[4] C. M. Bender, S. Boettcher, and M. Moshe, J. Math. Phys. **35**, 4941 (1994).

[5] C. Itzykson and J.-M. Drouffe, *Statistical Field Theory* (Cambridge University Press, Cambridge, 1989), Vol. 1.

[6] *Handbook of Mathematical Functions*, Natl. Bur. Stand. Appl. Math. Ser. No. 55, edited by M. Abramowitz and I. A. Stegun (U.S. GPO, Washington, DC, 1964), Chap. 22.

[7] A. Erdélyi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. II, Chap. X.

[8] See, M. Aizenman, Phys. Rev. Lett. **47**, 1 (1981); Commun. Math. Phys. **86**, 1 (1982); J. Frölich, Nucl. Phys. B **200**, 281 (1982); in *Progress in Gauge Field Theory*, edited by G. 't Hooft, A. Jaffe, H. Lehmann, P. K. Mitter, I. M. Singer, and A. Stora (Plenum, New York, 1984), p. 169.

[9] S. Boettcher and M. Moshe, Phys. Rev. Lett. **74**, 2410 (1995); S. Boettcher, Phys. Rev. E **51**, 3862 (1995).

[10] C. M. Bender, S. Boettcher, and P. N. Meisinger, following paper, Phys. Rev. E **53**, 112 (1996).

[11] C. M. Bender, S. Boettcher, and P. N. Meisinger, this issue, Phys. Rev. E **53**, 127 (1996).

[12] W. Lederman and G. E. H. Reuter, Philos. Trans. R. Soc. London Ser. A **246**, 321 (1954).

[13] The result in Eq. (4.5) at $m=1$ can be rederived using the continued-fraction methods of Ref. [1]. We simply substitute

$$Q_n = P_{\rm out}(n) P_{\rm in}(n+1) = \frac{n(n+D-2)}{(2n+D-3)(2n+D-1)}$$

into Eq. (3.21) of Ref. [1]. We obtain a continued-fraction representation for $\Pi_1(D)$, which can be simplified to read

$$\Pi_1(D) = 1/[D+1-2(D)/(D+3-3(D+1)/\{D+5-4 \times(D+2)/[D+7-5(D+3)/(\cdots)]\})].$$

If this continued fraction is truncated before the factors (D), $(D+1)$, $(D+2)$, $(D+3)$, and so on, each successive truncation gives exactly $\Pi_1(D) = 1/(D-1)$.

[14] C. M. Bender, S. Boettcher, and L. Lipatov, Phys. Rev. Lett. **68**, 3674 (1992).

[15] C. M. Bender, S. Boettcher, and L. Lipatov, Phys. Rev. D **46**, 5557 (1992).

[16] C. M. Bender and S. Boettcher, Phys. Rev. D **48**, 4919 (1993).

[17] C. M. Bender and K. A. Milton, Phys. Rev. D **50**, 6547 (1994).

[18] The one-dimensional case of this problem was investigated by T. E. Clark, R. Menikoff, and D. H. Sharp, Phys. Rev. D **22**, 3012 (1980).

[19] I. S. Gradsteyn and I. M. Ryzhik, in *Table of Integrals, Series and Products*, edited by A. Jeffrey (Academic, New York, 1980).